STAT GR 5203 PROBABILITY
LECTURE 1

Probability Theory: "mathematically encode uncertainty"

e.g., will it rain tomorrow? There's a 50% chance of it raining tomorrow

encodes our uncertainty about whether it will rain tomorrow

How do we make this rigorous?

1) Classical interpretation: "everything that can occur is equally likely" (limited)
2) Long run interpretation: "what is the long run result of conducting many trials"? (limited)
3) This class: axiomatic approach

To introduce these axioms, we need to define:

- Sample space
- Event

} (1.2)

We also need some set theory } (1.3?)

1.2 Sample space + event

DEF: An *experiment* is a process where we can identify the outcomes ahead of time

Ex. tossing a coin, rolling a die, using a battery

$S=\{H,T\}$ $\quad S=\{1,\dots,6\}$ $\quad S=\{x: 0\le x<\infty\}$

DEF: The *sample space*, S, is a set of all possible outcomes of an experiment

Ex. see above

(*) Ex. Experiment is tossing two coins

$S=\{HH, HT, TH, TT\}$

Experiment is rolling two die

$S=\{(i,j): i=1,\dots,6 \text{ and } j=1,\dots,6\}$

$$= \left\{ \begin{matrix} (1,1), & (1,2), & \dots \\ (2,1) & (2,2) & \dots \\ & & \ddots \end{matrix} \right\}$$

E

DEF: An *event* is a (well-defined) set of possible outcomes (i.e., a subset of S)

(*)

Ex. Experiment is tossing two coins

Event is getting at least one head

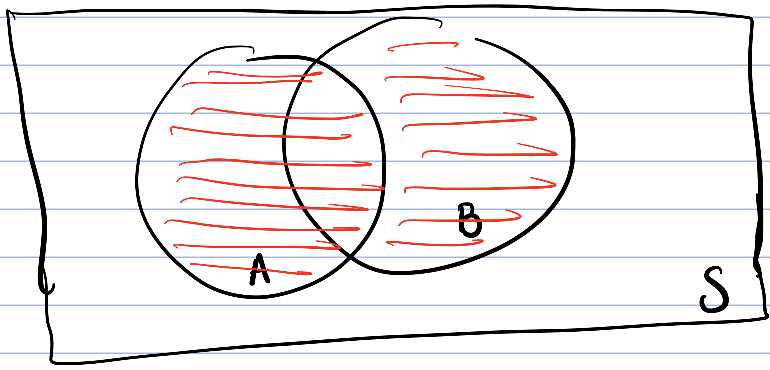
$E=\{HH, HT, TH\}$ $\quad E\subset S$

Ex. Experiment is rolling two die
Event is that the sum of the two die is ≤ 3
$E = \{(1,1), (1,2), (2,1)\}$

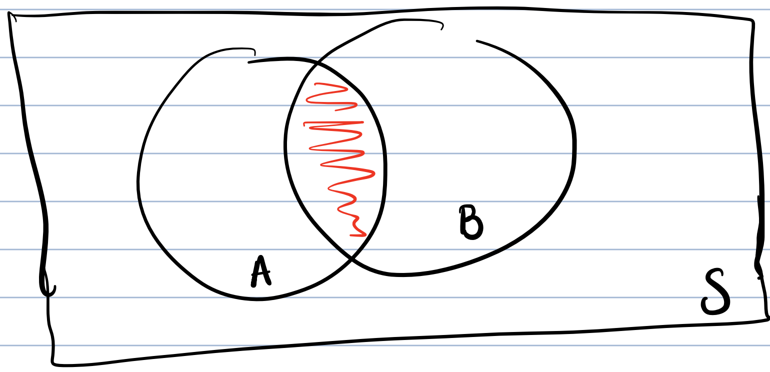
1.3 Set Theory : Consider two sets A, B where $A \subseteq S$ and $B \subseteq S$

SET OPERATIONS

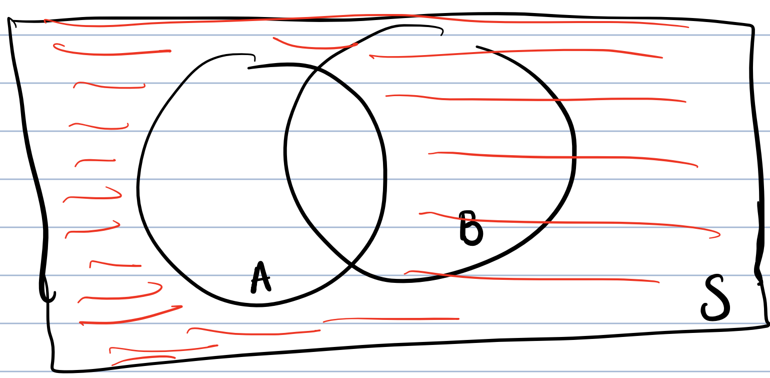
1) Union : $A \cup B = \{x \in S : x \in A \text{ OR } x \in B\}$



2) Intersection : $A \cap B = \{x \in S : x \in A \text{ AND } x \in B\}$
(AB)

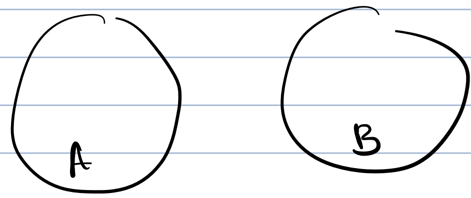


3) Complement : $A^c = \{x \in S : x \notin A\}$



SET RELATIONSHIPS

1) Containment : $A \subset B$ if $x \in A \Rightarrow x \in B$
2) Disjoint / mutually exclusive : A and B are disjoint (or mutually exclusive) if $A \cap B = \phi$
↳ The empty set



Example : Experiment is selecting a card randomly from a 52 card deck and we look at its suit

$S = \{C, Sp, H, D\}$

Example : $A = \{C, Sp\}$ $B = \{C, D\}$ $E = \{H\}$

$A \cup B = \{C, Sp, D\}$
$A \cap B = \{C\}$
$A \cap E = \phi$
$B^c = \{H, Sp\}$

$A = B$ if $x \in A \Rightarrow x \in B$
and $x \in B \Rightarrow x \in A$

Set laws : Consider sets E, F, G

1) Commutativity: $E \cup F = F \cup E$, $EF = FE$ $EF = E \cap F$
2) Associativity: $(E \cup F) \cup G = E \cup (F \cup G)$
$(EF)G = E(FG)$
3) Distributive: $(E \cup F)G = EG \cup EF$
$(EF) \cup G = (E \cup G)(F \cup G) = (E \cup G) \cap (F \cup G)$
4) DeMorgan's laws : $(E \cap F)^c = E^c \cup F^c$
$(E \cup F)^c = E^c \cap F^c$

More than two events : Consider events $A_1, A_2, A_3, \dots, A_n$ (Can be an infinite collection of events)

$\bigcap_{i=1}^{n} A_i = \{x \in S : x \in A_1 \text{ AND } x \in A_2, \dots, x \in A_n\}$

$\bigcup_{i=1}^{n} A_i = \{x \in S : x \in A_1 \text{ OR } x \in A_2, \dots, x \in A_n\}$

☆ : Generalized Demorgan's laws ; $\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$

$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$

Axioms of probability : Experiment with sample space S. Assume $\mathbb{P}(E)$, where $E \subseteq S$, is a well-defined object. For all $E \subseteq S$,
1. $0 \le \mathbb{P}(E) \le 1$
2. $\mathbb{P}(S) = 1$
3. For any sequence of disjoint events $E_1, E_2, \dots$ we have that
$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$

Facts about $\mathbb{P}(E)$ using the axioms:

① $\mathbb{P}(\phi) = 0$

$S^c = \phi$

$\mathbb{P}(S) = 1$

$\mathbb{P}(\phi) = \mathbb{P}(S^c) = 1 - \mathbb{P}(S) = 1 - 1 = 0$

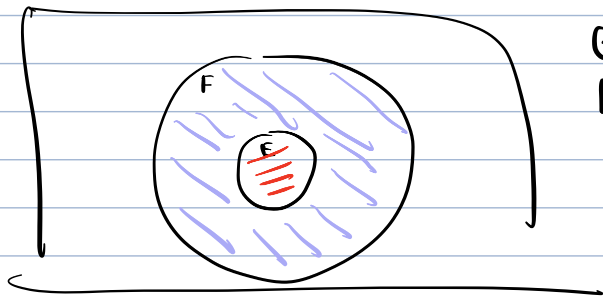
$$S = S \cup \phi = \mathbb{P}(S) \underset{(A3)}{=} \mathbb{P}(S) + \mathbb{P}(\phi)$$

$$\Rightarrow 1 = 1 + \mathbb{P}(\phi)$$

$$\Rightarrow 0 = \mathbb{P}(\phi)$$

② If $E \subseteq F$. Then $\mathbb{P}(E) \leq \mathbb{P}(F)$

• $E \subseteq F$ means $E \subset F$ or $E = F$



$G = E^c F$ " $F - E$ "

$F = E \cup G$

• $E \subset F$ means $x \in E \Rightarrow x \in F$ but $x \in F \not\Rightarrow x \in E$

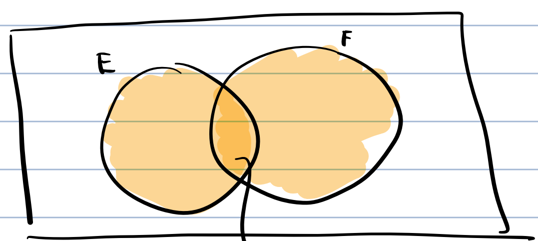
$$(A3) \Rightarrow \mathbb{P}(F) = \mathbb{P}(E \cup G) = \mathbb{P}(E) + \underbrace{\mathbb{P}(E^c F)}_{\geq 0 \ (A1)}$$

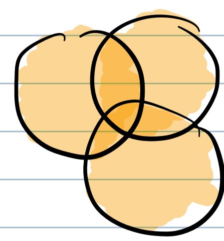
$$\geq \mathbb{P}(E) \checkmark$$

Principle of inclusion and exclusion: We can use (A1) - (A3) to show the following:

For $E, F \subseteq S$ $\quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

<u>Proof</u>: HW





$\mathbb{P}(E \cap F)$

$$\mathbb{P}(E \cup F \cup G) = \mathbb{P}(E) + \mathbb{P}(F) + \mathbb{P}(G) - \mathbb{P}(EF) - \mathbb{P}(EG) - \mathbb{P}(FG) + \mathbb{P}(EFG)$$

Ex. There are two possible diseases: bacterial infection (B)
viral infection (V)

$\mathbb{P}(B) = 0.7$; $\mathbb{P}(V) = 0.4$; $\mathbb{P}(B \cup V) = 1$

$\mathbb{P}(B \cap V) = ?$ $\qquad \mathbb{P}(B \cup V) = 1 = \mathbb{P}(B) + \mathbb{P}(V) - \mathbb{P}(B \cap V)$

$$1 = 0.7 + 0.4 - \mathbb{P}(B \cap V)$$

$$\Rightarrow \mathbb{P}(B \cap V) = 0.1$$

Simple sample spaces : sample spaces where the number of elements is finite + the outcomes are equally likely

Ex. $\{H, T\}$ Tossing 1 coin

$\{HH, HT, TH, TT\}$ 2 coins

$\{1, 2, 3, 4, 5, 6\}$ Rolling a dice

Ex. ① Toss 2 coins. $\mathbb{P}(HH) = ? = \frac{1}{4} = \frac{\text{\# of outcomes with HH}}{\text{total \# of outcomes}}$

$\mathbb{P}(\text{at least one } H) = \frac{3}{4} = \frac{\text{\# of outcomes with } \geq 1H}{\text{total \# of outcomes}}$

$\mathbb{P}(\text{only one } H) = \frac{2}{4} = \frac{\text{\# of outcomes with 1 H}}{\text{total \# of outcomes}} = \frac{1}{2}$

$\mathbb{P}(E) = \frac{\text{\# of outcomes with E}}{\text{total \# of outcomes}}$ we will introduce counting techniques to compute these

Counting (Combinatorics)

Basic rules of Counting

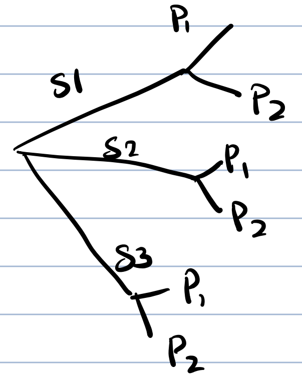
1. Sum rule : If one task can be done m ways
another task can be done n ways
and if they cannot be done at the same time
total # of ways to do both $= m+n$ ways

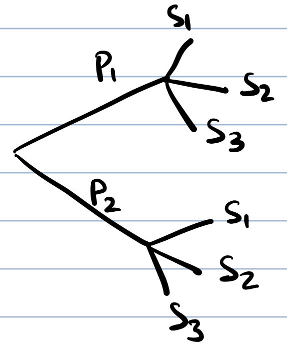
e.g., 10 statistics books and 5 math books
15 (10+5) books to choose from

2. Multiplication rule : If there are two successive tasks and there are m ways to do task 1 and n ways to do task 2, then there are $m \times n$ ways to task 1 then task 2

e.g., you have 3 shirts and 2 pants. How many outfits?

2×3





This generalizes : $n_1 \times n_2 \times \dots \times n_r$ r choices
n_i ways $i = 1, \dots, r$

Permutations: Say we have a set $A = \{a_1, a_2, a_3, \dots, a_n\}$

DEF: A permutation is an ordered arrangement of the elements of A

e.g., $A = \{C, A, R\}$

RAC, ARC, ACR, ... ← permutations of A

How many permutations are there of A? e.g., 3!

$\underline{\quad} \; \underline{\quad} \; \underline{\quad}$

$3 \times 2 \times 1 = 3!$

DEF: A factorial (!) is defined as follows: for an integer $n \geq 0$

$n! = n \times (n-1) \times (n-2) \times \dots \times 1$

Fact: $0! = 1$

★ The number of permutations of n elements of a set is given by $n!$

What if we wanted to only permute some elements of A?

Say we have a set $A = \{a_1, a_2, a_3, \dots, a_n\}$. How many permutations are there of r $(r < n)$ elements of A?

e.g., Consider a group of 5 people. How many ways are there to select a President, VP, and a secretary?

$\underset{P}{\underline{\quad}} \quad \underset{VP}{\underline{\quad}} \quad \underset{S}{\underline{\quad}}$

$5 \times 4 \times 3 = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}} = \frac{5!}{(5-3)!}$

★ The number of permutations of r $(r<n)$ elements from a set of n elements is given by $\frac{n!}{(n-r)!}$

e.g., Consider a group of 5 people. How many ways are there to select a President, VP, and a secretary if person A will only serve if they are president?

A serves $\quad \underset{P}{\underline{1}} \times \underset{VP}{\underline{4}} \times \underset{S}{\underline{3}} = 12$

A does not serve $\quad \underset{P}{\underline{4}} \times \underset{VP}{\underline{3}} \times \underset{S}{\underline{2}} = 24$

$12 + 24 = 36$