Last time

- Intro to probability + Kolmogorov's axioms
- Simple sample spaces
- Counting (sum rule, multiplication rule, permutations without repeats)

Today

- Counting cont. (permutations w/ repeats, Combinations, binomial/multinomial coefficients)
- Conditional probability
- Law of total probability
- Bayes rule (hopefully)

Tentative OH : Mondays 5:30 - 6:30
Thursdays 4-5

Location : TBD

Counting (cont.)

Permutations : # of ways to order n elements of a set : $n!$

# of ways to order r of those n elements : $\frac{n!}{(n-r)!}$

e.g., Group of 5 people. How many ways are there to choose a president, VP, and a secretary?

$$\underset{5}{\overline{P}} \times \underset{4}{\overline{VP}} \times \underset{3}{\overline{S}} = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}}$$

Permutations w/ repeated elements :

e.g., How many permutations are there of the word MOM $\frac{3!}{2!\,1!} = \frac{3!}{2!}$

$n=3$
$n_1=2$

Suppose the elements are $\{M_1, O, M_2\}$ 3! permutations of this set

$M_1 M_2 O$ $\quad$ $M_2 M_1 O$

$M_2 O M_1$ $\quad$ $M_1 O M_2$

$O M_2 M_1$ $\quad$ $O M_1 M_2$

$\frac{3!}{2!}$ ← overcounting by the number of permutations of M_1 and M_2

Generally : If a set has n elements and n_1 are the same $(n_1 \leq n)$

n_2 are the same $(n_2 \leq n)$

$\vdots$

n_r are the same $(n_r \leq n)$

Then the number of permutations is given by $\frac{n!}{n_1!\,n_2! \times \cdots \times n_r!}$

e.g., How many permutations are there of the word MISSISSIPPI $\frac{11!}{4!\,4!\,2!\,1!}$

$\hookrightarrow n = 11$

$n_1 = 4 \quad (4\ s)$

$n_2 = 4 \quad (4\ i)$

$n_3 = 2 \quad (2\ P)$

$n_4 = 1 \quad (1\ M)$

Combinations: number of unordered arrangements of r elements from a set of n elements

e.g., Permutation: 5 people, selecting a P, VP, S
Combination: 5 people, selecting 3 board members

$$\frac{5!}{2!\,\boxed{}} = \frac{5!}{3!\,2!}$$

$\hookrightarrow$ 3! the number of ways to order the 3 selected people

In general, the # of ways to choose (unordered) r elements from a set of n elements is $\frac{n!}{r!\,(n-r)!}$ OR # permutations = # combinations × r!

★ Combinations special case of repeat permutations

e.g., board members
C = chosen
N = not chosen

$$\frac{5!}{3!\,2!} = \text{\# of permutations of CCCNN}$$

Notation: $\frac{n!}{r!\,(n-r)!} \equiv \underbrace{\binom{n}{r}}_{\text{binomial coefficient}} \equiv$ "n choose r"

e.g., A person has 8 friends, 5 of whom will be invited to a party
(a) How many choices are there if 2 friends are feuding and will not attend together?

Approach 1

Scenario 1: neither attends $\binom{6}{5}$

Scenario 2: one feuding friend attends $\binom{2}{1} \times \binom{6}{4}$

$(2 \times \binom{6}{4})$

Total: $\binom{6}{5} + \binom{2}{1}\binom{6}{4}$

Approach 2

$\binom{8}{5} - \binom{6}{3}$

(b) How many choices are there if 2 friends will only attend together?

Scenario 1: both attend together $\binom{6}{3}$

Scenario 2: neither attend $\binom{6}{5}$

Total: $\binom{6}{3} + \binom{6}{5}$

Binomial and Multinomial coefficients

Recall: $\frac{n!}{r!(n-r)!} = \binom{n}{r}$ (Binomial coefficient)

$$\frac{n!}{n_1!\, n_2! \times \dots \times n_r!} = \binom{n}{n_1, n_2, \dots, n_r} \quad (n_1 + n_2 + \dots + n_2 = n)$$

(Multinomial coefficient)

e.g., 5 people, choosing 3 board members
$\equiv$ how to divide up 5 people into groups of 2 and 3

Binomial case: $\binom{5}{2} = \frac{5!}{2!3!}$

e.g., 10 police officers choose 5 for patrol (P)
2 for office (O)
3 reserve (R)

Multinomial case

\# of arrangements of PPPPPOORRR $\frac{10!}{5!2!3!} = \binom{10}{5,2,3}$

$\equiv \binom{10}{5} \times \binom{5}{2} \times \binom{3}{3}$

e.g., 20 club members choose 8 for Cmt 1
8 for Cm 2
4 for Cmt 3

$\frac{20!}{8!8!4!} = \binom{20}{8,8,4}$

Examples (computing probabilities by counting)

① Two cards are chosen from a deck of 52

(a) $\mathbb{P}$(both cards are aces) $= \frac{\text{\# of ways to choose 2 aces}}{\text{\# of ways to choose 2 cards from a 52 card deck}}$

$$= \frac{\binom{4}{2}}{\binom{52}{2}}$$

(b) $\mathbb{P}$(both cards are spades) $= \frac{\binom{13}{2}}{\binom{52}{2}}$

② 5 cards are chosen from a deck of 52

$\mathbb{P}$(4 aces in a 5 card hand) $= \frac{\text{\# of 5 card hands where 4 cards are aces}}{\text{\# of ways to choose 5 cards from 52}}$

A A A A __ (48)

$\binom{4}{4} \times \binom{48}{1}$

$$= \frac{48}{\binom{52}{5}}$$

A A A _ _

$\binom{4}{3}\binom{48}{2}$

(3) 4 players A, B, C, D each get 13 cards

$\mathbb{P}$(A gets 6 ♡, B gets 4 ♡, C gets 2 ♡, D gets 1 ♡)

$$= \frac{\text{'' \qquad '' but with A-6, B-4, C-2, D-1 ♡}}{\text{\# of ways to divide up 52 cards into 4 groups of 13}}$$

$$= \binom{13}{6,4,2,1}$$

$$= \frac{\underbrace{\binom{13}{6}\binom{7}{4}\binom{3}{2}\binom{1}{1}}_{''} \times \binom{39}{7,9,11,12}}{\binom{52}{13,13,13,13}}$$

$$\binom{52}{13,13,13,13} \equiv \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

★ Need to go over the birthday problem

Ch 2: Conditional probability

Conditional prob: restricts the sample space

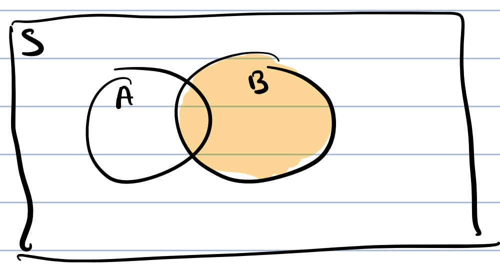
" What is the probability of B occuring given that A has occured? "

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} \leftarrow \mathbb{P}(A) > 0$$

↑ given

Unconditional prob:

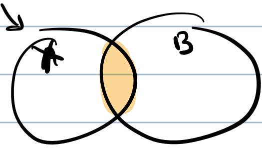
$$\mathbb{P}(B) = \frac{\text{Area of } B}{\text{Area of } S}$$



conditional prob:

$$\mathbb{P}(B \mid A) = \frac{\text{Area of } A \cap B}{\text{Area of } A}$$

new S



Examples

① Suppose we flip 2 coins. What probability of HH given that there is at least one head?

$S = \{HH, HT, TH, TT\}$

$\geq H = \{HH, HT, TH\}$

$$\mathbb{P}(HH \mid \geq 1 H) = \frac{\mathbb{P}(HH \cap \geq 1 H)}{\mathbb{P}(\geq 1 H)} = \frac{\mathbb{P}(HH)}{\mathbb{P}(\geq 1 H)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Using: If $A \subset B$, then $\mathbb{P}(A \cap B) = \mathbb{P}(A)$

② Suppose we roll a die

$$\mathbb{P}(\text{roll being even} \mid \text{roll} \leq 3) = 1/3$$

$\text{roll} \leq 3 = \{1, 2, 3\}$

$1/3$

$$= \frac{\mathbb{P}(\text{roll being even} \cap \text{roll} \leq 3)}{\mathbb{P}(\text{roll} \leq 3)} = \frac{1/6}{3/6}$$

③ A family has 2 kids. $\mathbb{P}(BB \mid \geq 1 B) = 1/3$

$\mathbb{P}(BB \mid 2^{\text{ND}} \text{ kid is a boy}) = \frac{1}{2}$

$S = \{BB, BG, GB, GG\}$

$2^{\text{ND}} \text{ boy} = \{BB, GB\}$

Facts about Conditional probs

① By definition $\mathbb{P}(B \mid A) = \dfrac{\mathbb{P}(AB)}{\mathbb{P}(A)} \Rightarrow \mathbb{P}(AB) = \mathbb{P}(B \mid A)\mathbb{P}(A)$

Conditional and intersection

$= \mathbb{P}(A \mid B)\mathbb{P}(B)$

② Law of total probability   Conditional and unconditional probs

$A = (A \cap B) \cup (A \cap B^c)$

(A3) $\Rightarrow \mathbb{P}(A) = \mathbb{P}(AB) + \mathbb{P}(AB^c)$

① $\Rightarrow \mathbb{P}(A) = \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\mathbb{P}(B^c)$

$B \cap B^c = \phi$

$B \cup B^c = S$

③ The above generalize to more than 2 events

① events $A_1, A_2, \ldots, A_n$

$\mathbb{P}(A_1 A_2 \ldots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_2 A_1) \times \cdots \times \mathbb{P}(A_n \mid A_1, \ldots, A_{n-1})$

(mutually exclusive)

② If $B_1, \ldots, B_n$ are disjoint events s.t. $\bigcup_{i=1}^{n} B_i = S$

Then $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)$

e.g., A factory has 3 machines A, B, C D = defective

A makes defectives w/p 5 %

B " " 4 %

C " " 2 %

A makes 25 % of the products

B 35%

C 40%

$\mathbb{P}$(selecting a defective product) $= 0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02$

$= \mathbb{P}(A)\mathbb{P}(D|A) + \mathbb{P}(B)\mathbb{P}(D|B) + \mathbb{P}(C)\mathbb{P}(D|C)$