# STAT GR5203 - Lecture 5

- Monday OH 10-11 AM (room TBD)
- Midterm policies + coverage posted
- Practice midterm will be visible at 3:40
- Review session?
- STAT help room

## Recap

- Intro to r.v.s and distributions
- pmf / pdf, cdf
- functions of r.v.s

**THM**: Suppose $X$ is continuous and $g(X)$ is $\underbrace{\text{strictly monotone}}_{\Rightarrow\, g \text{ is one-to-one}}$. Then $Y = g(X)$ has pdf given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

$g^{-1}(y)$ is the value of $x$ s.t. $g(x) = y$

Overall idea: If $g$ is one to one

$$\mathbb{P}(g(X) \leq c) = \mathbb{P}(X \leq g^{-1}(c))$$

← if $X$ is discrete then use pmf

If $X$ is continuous you can find the pdf by differentiation

## Examples

1. $X$ is a continuous r.v., with pdf $f_X$. What is the pdf of $Y = X^2$?
   $X$ is supported on $\mathbb{R}$

1. Compute cdf of $Y$: for $y \geq 0$ (because $y = x^2 \geq 0$)

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\sqrt{X^2} \leq \sqrt{y}) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & \text{for } y \geq 0 \end{cases}$$

2. $y \geq 0$, $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$

$$= \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y})$$

Chain rule

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot -\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

$y < 0$, $f_Y(y) = 0$

2. Let $X$ be a continuous, nonnegative r.v. with pdf $f_X$. Let $Y = X^n$. What is the pdf of $Y$? You can use THM or the cdf method

$\Rightarrow g$ is one-to-one

THM (Mon.): Suppose $X$ is continuous and $g(X)$ is strictly monotone. Then $Y = g(X)$ has pdf given by

(Example 2 $Y = X^n$ for $x \geq 0$)

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

$g^{-1}(y)$ is the value of $x$ s.t. $g(x) = y$

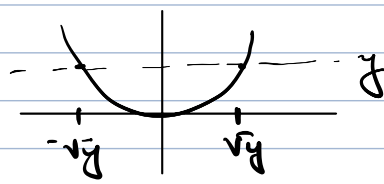
THM (non-Mon.): Suppose $X$ is continuous Then $Y = g(X)$ has pdf given by

(Example 1 $Y = X^2$ for $X \in \mathbb{R}$)

$$f_Y(y) = \begin{cases} \sum_{x:\, g(x)=y} f_X(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

$g^{-1}(y)$ is the value of $x$ s.t. $g(x) = y$

e.g., $Y = X^2$




$$f_Y(y) = \frac{1}{2\sqrt{y}} \left(f_X(-\sqrt{y}) + f_X(\sqrt{y})\right)$$

Probability integral transform (useful for generating random samples from a distribution)

THM: Let $X$ be a continuous r.v. with cdf $F_X$, and let $Y = F_X(X)$. Then $Y \sim \text{Unif}(0,1)$

" has the distribution" or "is distributed as"

strictly monotone (general case is in Degroot)

Proof: Find the cdf of $y$: for $y \in [0,1]$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(F_X(X) \leq y) = \mathbb{P}(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y & y \in [0,1] \\ 1 & y > 1 \end{cases}$$

cdf of uniform!

(Implication of the above)

Corollary: If $Y \sim \text{Unif}(0,1)$ and $F$ is a continuous cdf with quantile function $F^{-1}$, Then $X = F^{-1}(Y)$ has cdf $F$

X want x to have pdf

e.g., How do you generate a random sample ~~from~~ with a distribution with pdf $f(x) = e^{-x}$ for $x > 0$?

1. Generate $Y \sim \text{Unif}(0,1)$
2. Find $F^{-1}(y) = -\log(1-y)$
3. $-\log(1-Y)$ has cdf $F$ (Corollary)

Indicator function

$$\mathbb{1}\{x > 0\} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{o/w} \end{cases}$$

How do we find $F^{-1}$?
1. Find $F$
2. Solve for $F^{-1}$

$$F_X(x) = \int_0^x e^{-x}\,dx = (1-e^{-x})\mathbb{1}\{x>0\}$$

$$y = F_X(x) = 1-e^{-x} \qquad (x>0)$$

$$e^{-x} = 1-y \Rightarrow x = -\log(1-y)$$
$$F^{-1}(y) = -\log(1-y) \qquad y \in (0,1)$$

## Multivariate random variables

Recall: Univariate r.v. $X: S \to \mathbb{R}$
Multivariate r.v. $X: S \to \mathbb{R}^n$
Special case: Bivariate r.v. $X: S \to \mathbb{R}^2$

e.g., Roll 2 dice. $X$ = sum of the two rolls
$Y$ = absolute difference between the two rolls
$S = \{(i,j) : i,j = 1,\dots,6\}$

$X$ can be $2,3,\dots,12$
$Y$ can be $0,1,\dots,5$

$$\mathbb{P}(X=4,\ Y=0) = \mathbb{P}(\{(x,y)\in S \text{ s.t. } x+y=4,\ |x-y|=0\})$$
$$= \mathbb{P}(\{2,2\}) = \frac{1}{36}$$

## Bivariate cdf

DEF: If $(X,Y)$ is a r.v. then the joint cdf $F_{XY}(x,y)$ is defined as

$$F_{XY}(x,y) = \mathbb{P}(X \le x,\ Y \le y) \text{ for } x,y \in \mathbb{R}$$
(compared to univariate $F_X(x) = \mathbb{P}(X \le x)$)

Properties of joint cdf:

univar

1) $F_{XY}(\infty,\infty) = \lim_{\substack{x\to\infty \\ y\to\infty}} F_{XY}(x,y) = 1$ $\quad$ $\left(F_X(\infty) = \lim_{x\to\infty} F_X(x) = 1\right)$

2) $F_{XY}(-\infty,y) = \lim_{x\to-\infty} F_{XY}(x,y) = 0$ $\quad$ $\left(F_X(-\infty) = \lim_{x\to-\infty} F_X(x) = 0\right)$

$F_{XY}(x,-\infty) = \lim_{y\to-\infty} F_{XY}(x,y) = 0$

NEW

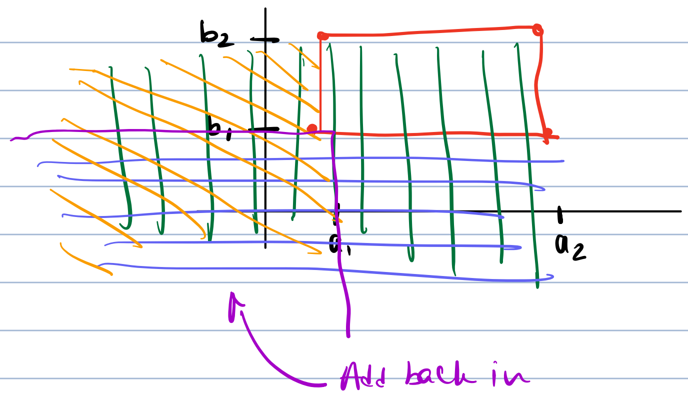
3) $F_{XY}(x,\infty) = \lim_{y\to\infty} F_{XY}(x,y) = F_X(x)$ $\leftarrow$ marginal cdf of $X$

$F_{XY}(\infty,y) = \lim_{x\to\infty} F_{XY}(x,y) = F_Y(y)$ $\leftarrow$ marginal cdf of $Y$

$3 \to 1$ $\quad F_{XY}(\infty,\infty) = \lim_{x\to\infty}\left(\lim_{y\to\infty} F_{XY}(x,y)\right) = \lim_{x\to\infty} F_X(x) = 1$

We can answer questions like

$$\mathbb{P}(a_1 \le X \le a_2,\ b_1 \le Y \le b_2) = F(a_2, b_2) - F(a_2, b_1) - F(a_1, b_2) + F(a_1, b_1)$$




## Discrete joint r.v.s

<u>DEF</u>: If $X$ and $Y$ are both discrete. Then we can define the joint pmf of $X$ and $Y$ as follows:

$$p_{XY}(x,y) = \mathbb{P}(X = x, Y = y) \quad \text{for } x, y \in \mathbb{R}$$

Note: The marginal pmf of $X$ can be obtained from $p_{XY}$ by "summing over $y$":

$$p_X(x) = \mathbb{P}(X = x) = \sum_{y:\, p_{XY}(x,y) > 0} p_{XY}(x,y)$$

Similarly; $p_Y(y) = \mathbb{P}(Y = y) = \sum_{x:\, p_{XY}(x,y) > 0} p_{XY}(x,y)$

Marginal Distributions

<u>Example</u>: Suppose an urn contains 3W, 2R, 5B

$X = \#$ of W

$Y = \#$ of B

We choose 2 balls from the urn

What is $p_{XY}$? What is $p_X$?

$X$ and $Y$ can be $\{0, 1, 2\}$ + $X + Y \le 2$

Write out $p(0,0)$, $p(0,1)$, etc. $\longrightarrow$

1. Identify possible values of $(x,y)$
2. Calculate $p(1,1)$ etc.

$P(X = 0) = p(0,0) + p(0,1) + p(0,2)$

$P(X = 1) = p(1,0) + p(1,1) + p(1,2)$

$P(X = 2) = p(2,0) + p(2,1) + \cancel{p(2,2)} \to 0$

3. To find $p_X(x)$, we need to sum over $y$

$$\sum_y p_{XY}(x,y)$$

How to compute $p(x,y)$?

$$p(0,0) = \frac{\#\text{ of ways to get } 0W \text{ and } 0B}{\#\text{ of ways to choose 2 balls from 10}} = \frac{\binom{2}{2}}{\binom{10}{2}}$$

Choosing only from Red balls

$$p(0,1) = \frac{\#\text{ of } 0W\ 1B}{\binom{10}{2}} = \frac{\binom{5}{1}\binom{2}{1}}{\binom{10}{2}}$$

$p(2,1) = 0$

$p(2,2) = 0$

Continuous joint r.v.s

DEF: A random vector $(X,Y)$ is called continuous if for any $C \subset \mathbb{R}^2$, there is
"multivariate r.v"
some function $f(x,y)$ such that

$$\mathbb{P}((X,Y) \in C) = \int_C f(x,y)\,dx\,dy$$

where $f: \mathbb{R}^2 \to \mathbb{R}$

joint pdf of $(X,Y)$

Relationship between pdf and cdf:

1) $F(a,b) = \mathbb{P}(X \le a, Y \le b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\,dy\,dx$

$a, b \in \mathbb{R}$

2) $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$

Note: We can obtain the marginal pdfs from the joint pdf

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Furthermore, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

e.g., Suppose $(X,Y)$ have joint pdf $(c>0)$

HW: compute integrals

$$f(x,y) = \begin{cases} ce^{-x}e^{-2y} & \text{for } \begin{array}{l} 0<x<\infty \\ 0<y<\infty \end{array} \\ 0 & \text{ow} \end{cases}$$

(a) What is $c$?

Set $\int_0^\infty \int_0^\infty ce^{-x}e^{-2y}\,dy\,dx = 1$ and solve for $c=2$

(b) $\mathbb{P}(X>1, Y<1)$

$\int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy$

(c) $\mathbb{P}(X<Y)$ (HW)

$\int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dx\,dy$ (*) $\leftarrow \iint_{x<y} 2e^{-x}e^{-2y}\,dx\,dy$

(d) $\mathbb{P}(X<a)$ for $a>0$?

1. $f_X(x) = \int_0^\infty 2e^{-x}e^{-2y}\,dy$

2. $\int_0^a f_X(x)\,dx$

## Functions of bivariate r.v.s

- Suppose $(X_1, X_2)$ is a continuous random vector with pdf $f_{X_1X_2}(a,b)$
- Suppose that pdf is supported on $A \subset \mathbb{R}^2 \Rightarrow \mathbb{P}((X_1,X_2)\in S) = 1$
- Suppose we apply functions $g_1$ and $g_2$ to $(X_1,X_2)$ satisfying the following

$$\begin{cases} y_1 = g_1(x_1,x_2) \\ y_2 = g_2(x_1,x_2) \end{cases} \quad (*)$$

(1) The eq. in (*) can be solved uniquely for $x_1$ and $x_2$ in $S$ in terms of $y_1$ and $y_2$. Denote the solutions by $x_1 = h_1(y_1,y_2)$, $x_2 = h_2(y_1,y_2)$ (Think of this as $g_1^{-1}, g_2^{-1}\ldots$)

(2) $g_1$ and $g_2$ have continuous partial derivatives for all $(x_1,x_2)\in S$ and are such that

determinant

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_2}{\partial x_1}\frac{\partial g_1}{\partial x_2} \neq 0 \quad \forall (x_1,x_2)\in S$$

Then, $(Y_1,Y_2) = (g_1(X_1,X_2), g_2(X_1,X_2))$ has the following pdf:

$$f_{Y_1Y_2}(y_1,y_2) = f_{X_1X_2}(h_1(y_1,y_2), h_2(y_1,y_2))\,|J(h_1(y_1,y_2), h_2(y_1,y_2))|^{-1}$$

for $(y_1,y_2)\in T$ where $T = (g_1,g_2)(S) \subset \mathbb{R}^2$ is the image of $g_1, g_2$

e.g., Let $(X_1, X_2)$ be continuous with pdf $f_{X_1X_2}(x_1, x_2)$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$
find $f_{Y_1Y_2}(y_1, y_2)$

① Find $h_1$, $h_2$

$h_1$: solve for $x_1$ $\quad$ $h_2$: solve for $x_2$

$$\begin{aligned} y_1 &= g_1(x_1, x_2) = x_1 + x_2 \\ + \quad y_2 &= g_2(x_1, x_2) = x_1 - x_2 \\ \hline y_1 + y_2 &= 2x_1 \end{aligned}$$

$$\Rightarrow x_1 = \frac{y_1 + y_2}{2} = h_1(y_1, y_2)$$

$$y_1 = x_1 + x_2 = \frac{y_1 + y_2}{2} + x_2$$

$$\rightarrow x_2 = y_1 - \frac{y_1 + y_2}{2} = \frac{y_1 - y_2}{2} = h_2(y_1, y_2)$$

② Compute J

$$\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(1) = -2$$

③ $f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) |J|^{-1}$

$$= f_{X_1X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right) \cdot \underbrace{|-2|^{-1}}_{=1/2} \quad \text{for } y_1, y_2 \in T$$

Suppose $f_{X_1X_2}(x_1, x_2) = \mathbb{1}\{0 < x_1 < 1\} \mathbb{1}\{0 < x_2 < 1\}$
Then from above, we have $\mathbb{1}\left\{0 < \frac{y_1 + y_2}{2} < 1\right\} \mathbb{1}\left\{0 < \frac{y_1 - y_2}{2} < 1\right\} = f_{Y_1Y_2}(y_1, y_2)$